REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 5
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): Detection in Incompletely Characterized Colored Non-Gaussian Noise via Parametric Modeling
5. TYPE OF REPORT & PERIOD COVERED: August 1984 to July 1986
6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Steven Kay and Debasis Sengupta
8. CONTRACT OR GRANT NUMBER(s): N00014-84-K-0527
9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Electrical Engineering Department
University of Rhode Island
Kingston, Rhode Island 02881
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS:
Office of Naval Research
Department of the Navy, Arlington, Virginia 22217
12. REPORT DATE: August 1986
13. NUMBER OF PAGES: 33
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report):
Approved for public release. Distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
Non-Gaussian Processes
Generalized Likelihood Ratio Tests
Uniformly Most Powerful Tests
Prewhitener-correlator
Autoregressive Processes
Signal to Noise Ratio
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):
The problem of detecting a signal known except for amplitude in incompletely characterized colored non-Gaussian noise is addressed. The problem is formulated as a testing of hypotheses using parametric models for the statistical behavior of the noise. A generalized likelihood ratio test is employed. It is shown that for a symmetric noise probability density function the detection performance is asymptotically equivalent to that obtained for a detector designed with a priori knowledge of the noise

Report Number 5

Detection in Incompletely Characterized Colored Non-Gaussian Noise via Parametric Modeling

cor

STEVEN KAY AND DEBASIS SENGUPTA

Department of Electrical Engineering
University of Rhode Island University
Kingston, Rhode Island 02881

August 1986

Prepared for

OFFICE OF NAVAL RESEARCH
Statistics and Probability Branch
Arlington, Virginia 22217
under Contract N00014-84-K-0527
S.M. Kay, Principal Investigator

Approved for public release; distribution unlimited

Abstract

The problem of detecting a signal known except for amplitude in incompletely characterized colored non-Gaussian noise is addressed. The problem is formulated as a testing of composite hypotheses using parametric models for the statistical behavior of the noise. A generalized likelihood ratio test is employed. It is shown that for a symmetric noise probability density function the detection performance is asymptotically equivalent to that obtained for a detector designed with *a priori* knowledge of the noise parameters. Non-Gaussian distributions of the noise are found to be more favorable for the purpose of detection as compared to the Gaussian distribution.

I. Introduction

The theory of detection of a known signal in presence of Gaussian noise having a known covariance matrix is well developed [Van Trees 1968]. In many applications, however, the covariance matrix is not known a priori. This difficulty can be alleviated by characterizing the correlation pattern of the noise by a simple model and using estimates of the model parameters to design a detector [Whalen 1971], [Bowyer *et al* 1979], [Kay 1983]. The difficulty increases when full information regarding the noise probability density function (PDF), usually assumed to be Gaussian, is unavailable due to insufficient knowledge about the noise source [Knight *et al* 1981]. There is no uniformly most powerful (UMP) test in this case because the use of a Neyman-Pearson criterion leads to a detector which depends on the unknown parameters. The Bayesian method of assigning priors to the unknown parameters of the noise PDF produces an 'optimal' detector [Lee *et al* 1977], but requires a multidimensional integration. Its performance is critically dependent on the accuracy of the choice of priors. A robust detector [Kassam and Poor 1985], on the other hand, does not use any partial knowledge about the noise PDF and therefore is not expected to perform well. Locally optimal (LO) detectors for this problem have been studied extensively by Czarnecki, Martinez, Thomas and others [Czarnecki and Thomas 1984], [Martinez and Thomas 1982]. Their results however rely on a known covariance matrix and marginal PDF of the noise. A third dimension is added to the problem if the amplitude of the signal is not known [Kay 1985]. A locally optimal detector can not be used since it depends on the polarity or sign of the amplitude, which is usually unknown.

This paper addresses the problem of detecting a deterministic signal known except for amplitude in the presence of incompletely characterized non-white non-Gaussian noise. The approach chosen here is to use the theory of the generalized likelihood ratio test (GLRT) for composite hypothesis testing [Kendall and Stuart 1979]. The work presented here is an extension of the work of Kay [1985] in which the noise is assumed

to be non-Gaussian but *white*. In this case the covariance matrix is assumed to be known except for a few parameters. Maximum likelihood estimates (MLE) for these parameters are then used in the GLRT. The asymptotic performance of the GLRT detector is shown to be equivalent to the asymptotic performance of the clairvoyant GLRT detector (one which uses perfect knowledge of the unknown parameters) for a symmetric noise PDF. Therefore the GLRT asymptotically achieves an upper bound in performance and is *optimal* in this sense.

The paper is organized as follows. Section II gives the theory of the GLRT which will be used extensively in the subsequent sections. Section III formulates the detection problem and derives the GLRT for it. The case of autoregressive (AR) noise is considered separately. Section IV discusses the performance of the GLRT detector and compares it to that of the clairvoyant GLRT detector. Section V draws some general conclusions about the performance of the GLRT . Section VI summarizes the results and discusses the implementation aspect of the problem.

II. Review of Generalized Likelihood Ratio Test

Consider the problem of testing the value of the parameter $\Theta = \left[\Theta_r^T \ \Theta_s^T\right]^T$ based on the a data set $\mathbf{y} = [y_1\ y_2\ \cdots\ y_N]$. Θ_r and Θ_s are assumed to be vectors of dimension r and s, respectively. A common hypothesis test is

$$\begin{aligned} \mathcal{H}_0 &: \Theta^T = \left[\mathbf{0}^T \ \Theta_s^T\right] \\ \mathcal{H}_1 &: \Theta^T = \left[\Theta_r^T \ \Theta_s^T\right] \qquad \Theta_r \neq \mathbf{0} \end{aligned} \tag{1}$$

Θ_s, referred to as the vector of nuisance parameters, is of no concern and may assume any value. Assuming the observed data $\mathbf{y}$ has a joint probability function $\mathbf{f}(\mathbf{y};\Theta_r,\Theta_s)$, a *generalized likelihood ratio test* for testing (1) is to decide $\mathcal{H}_1$ if

$$\ell_G = \frac{\mathbf{f}(\mathbf{y};\hat{\hat{\Theta}}_r,\hat{\hat{\Theta}}_s)}{\mathbf{f}(\mathbf{y};\mathbf{0},\hat{\Theta}_s)} > \gamma \tag{2}$$

for some threshold γ. $\mathbf{0}$ is an r-dimensional vector of zeroes. $\hat{\Theta}_s$ is the MLE of Θ_s assuming $\mathcal{H}_0$ is true while $\hat{\hat{\Theta}}_r$ and $\hat{\hat{\Theta}}_s$ are *joint* MLE's of Θ_r and Θ_s assuming $\mathcal{H}_1$ is

true. $\hat{\Theta}_s$ is found by maximizing $\mathbf{f}(\mathbf{y}; 0, \Theta_s)$ over Θ_s. Similarly, $\hat{\hat{\Theta}}_r$, $\hat{\hat{\Theta}}_s$ are obtained by maximizing $\mathbf{f}(\mathbf{y}; \Theta_r, \Theta_s)$ over Θ_r and Θ_s.

The statistics of ℓ_G are difficult to obtain in general. For large data records (asymptotically) it may be shown that $2 \ln \ell_G$ is distributed in the following manner [Kendall and Stuart 1979].

$$2 \ln \ell_G \sim \chi^2_r \qquad \text{under } \mathcal{H}_0 \tag{3a}$$

$$2 \ln \ell_G \sim \chi'^2(r, \lambda) \qquad \text{under } \mathcal{H}_1 \tag{3b}$$

Here χ^2_r represents a chi-square distribution with r degrees of freedom and $\chi'^2(r, \lambda)$ represents a noncentral chi-square distribution with r degrees of freedom and noncentrality parameter λ. Note that $\chi'^2(r, 0) = \chi^2_r$ or the distribution under $\mathcal{H}_0$ is a special case of the distribution under $\mathcal{H}_1$ and occurs when $\lambda = 0$. The noncentrality parameter λ, which is a measure of the discrimination between the two hypotheses, is given by

$$\lambda = \Theta_r^T \left[\mathbf{I}_{\Theta_r \Theta_r}(0, \Theta_s) - \mathbf{I}_{\Theta_r \Theta_s}(0, \Theta_s) \mathbf{I}^{-1}_{\Theta_s \Theta_s}(0, \Theta_s) \mathbf{I}^T_{\Theta_r \Theta_s}(0, \Theta_s) \right] \Theta_r \tag{4}$$

where Θ_r, Θ_s are the true values. The terms in the brackets of (4) are found by partitioning the Fisher information matrix for Θ as

$$\mathbf{I}(\Theta) = \begin{pmatrix} \mathbf{I}_{\Theta_r \Theta_r}(\Theta_r, \Theta_s) & \mathbf{I}_{\Theta_r \Theta_s}(\Theta_r, \Theta_s) \\ \mathbf{I}_{\Theta_s \Theta_r}(\Theta_r, \Theta_s) & \mathbf{I}_{\Theta_s \Theta_s}(\Theta_r, \Theta_s) \end{pmatrix} \tag{5}$$

and the partitions are defined as

$$\begin{aligned}
\mathbf{I}_{\Theta_r \Theta_r}(\Theta_r, \Theta_s) &= E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \Theta_r} \right) \left(\frac{\partial \ln \mathbf{f}}{\partial \Theta_r} \right)^T \right] && r \times r \\
\mathbf{I}_{\Theta_r \Theta_s}(\Theta_r, \Theta_s) &= E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \Theta_r} \right) \left(\frac{\partial \ln \mathbf{f}}{\partial \Theta_s} \right)^T \right] && r \times s \\
\mathbf{I}_{\Theta_s \Theta_r}(\Theta_r, \Theta_s) &= \mathbf{I}^T_{\Theta_r \Theta_s}(\Theta_r, \Theta_s) && s \times r \\
\mathbf{I}_{\Theta_s \Theta_s}(\Theta_r, \Theta_s) &= E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \Theta_s} \right) \left(\frac{\partial \ln \mathbf{f}}{\partial \Theta_s} \right)^T \right] && s \times s
\end{aligned} \tag{6}$$

All the partitions of the Fisher information matrix are evaluated at $\Theta_r = 0$ and the true values of Θ_s for use in (4).

The motivation for using a GLRT is that for large data records it exhibits certain optimality properties. A uniformly most powerful (UMP) test does not exist in many situations. However, of all the tests which are invariant to a natural set of transformations the GLRT exhibits the largest probability of detection. The GLRT is said to be the asymptotically *uniformly most powerful invariant* (UMPI) test [Lehmann 1959]. It is also a *consistent* test in the sense that the probability of deciding $\mathcal{H}_0$ when $\mathcal{H}_1$ is actually true approaches 0 for large data records. Asymptotically the GLRT is *unbiased*, *i.e.*, the probability of detection when $\mathcal{H}_1$ is true is larger than the probability of false alarm. (This result follows from (3) and properties of the chi-square distribution.) Finally, although the GLRT does not usually exhibit a *constant false alarm rate* (CFAR) it does so for large data records. It is difficult to find the conditions under which the asymptotic results apply to finite length data records. The following heuristic conditions follow from [Cox and Hinkley 1974].

1) The asymptotic statistics of the MLE's used in the likelihood ratio should be applicable, *i.e.*, they should be Gaussian with mean equal to the true parameter value and covariance matrix equal to the inverse of the Fisher information matrix.
2) The two hypotheses should be reasonably close and only slight departures of Θ_r from zero should be tested.

III. Formulation of the Problem and GLRT Solution

The detection problem considered here is the following.

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{y} = \mathbf{W}\mathbf{u} \\ \mathcal{H}_1 &: \mathbf{y} = \mathbf{W}\mathbf{u} + \mu\mathbf{s} \end{aligned} \tag{7}$$

where $\mathbf{s} = [s_1 \ s_2 \ \cdots \ s_N]^T$ is a vector of known signal amplitudes, $\mathbf{u} = [u_1 \ u_2 \ \cdots \ u_N]^T$

is a vector of *independent and identically distributed* (*i.i.d.*) noise with a symmetric PDF, μ is an unknown scalar (either positive or negative) and $\mathbf{W}$ is an invertible $(N \times N)$ matrix whose elements are functions of a set of unknown parameters $\Psi = [\psi_1 \ \psi_2 \ \cdots \ \psi_M]$.

$$[\mathbf{W}]_{ij} = \mathrm{w}_{ij}(\Psi)$$

Since $u_n, \quad n = 1, 2, \cdots N$ are *i.i.d.*, the PDF of $\mathbf{u}$ can be expressed as

$$\mathbf{f}(\mathbf{u}; \Phi) = \prod_{n=1}^{N} f(u_n; \Phi) \tag{8}$$

where $f(u_n; \Phi)$ is the marginal PDF of each u_n dependent on the unknown parameter vector Φ. f is assumed to be symmetric, *i.e.*, $f(-u) = f(u)$. Note that the covariance matrix of the noise is $\sigma^2 \mathbf{W}\mathbf{W}^T$ where σ^2 is the variance of u_n.

(7) represents a general set of problems. The unknown matrix $\mathbf{W}$ allows for a *large class of spectral characteristics or correlation patterns of the background noise.* For large data records autoregressive (AR), moving average (MA) and autoregressive moving average (ARMA) processes can be represented by the above formulation of the underlying random process if $\mathbf{W}$ is the impulse response matrix of the corresponding filter. Secondly, the PDF of u_n can be chosen to characterize specific problems in a realistic way. The parameter vector Φ is left unknown in order to *add flexibility to the noise PDF model.* Thirdly, by allowing μ to be positive or negative the detector will be able to *accommodate a change of polarity* in the signal.

The problem of (7) can be recast as

$$\mathcal{H}_0 : \Theta^T = \left[\mathbf{0}^T \ \Theta_s^T\right] \tag{9a}$$

$$\mathcal{H}_1 : \Theta^T = \left[\Theta_r^T \ \Theta_s^T\right] \qquad \Theta_r \neq \mathbf{0} \tag{9b}$$

where

$$\begin{aligned} \Theta_r &= \mu \qquad \text{a scalar} \\ \Theta_s &= [\Psi^T \ \Phi^T]^T \qquad \text{(vector of } \textit{nuisance} \text{ parameters)} \end{aligned} \tag{10}$$

Since (9) is equivalent to (1), the GLRT for testing $\mathcal{H}_1$ vs. $\mathcal{H}_0$ is given by (2). In order to evaluate the MLE's it is necessary to find the joint PDF of $\mathbf{y}$ under either hypothesis which can be found from the joint PDF of $\mathbf{u}$ in the following way. From (7) it follows that

$$\mathbf{u} = \mathbf{W}^{-1}\mathbf{y} \qquad \textit{under } \mathcal{H}_0 \tag{11a}$$

$$\mathbf{u} = \mathbf{W}^{-1}(\mathbf{y} - \mu\mathbf{s}) \qquad \textit{under } \mathcal{H}_1 \tag{11b}$$

$\mathbf{W}^{-1}$ exists because $\mathbf{W}$ is assumed to be invertible. The elements of $\mathbf{W}^{-1}$ are also known functions of Ψ.

$$\left[\mathbf{W}^{-1}\right]_{ij} = \omega_{ij}(\Psi) \tag{12}$$

(11) being an affine transformation, the joint PDF of $\mathbf{y}$ can be written as

$$\mathbf{f}(\mathbf{y}; \Psi, \Phi) = \frac{1}{|\det(\mathbf{W})|}\mathbf{f}(\mathbf{u}; \Phi)\Big|_{\mathbf{u}=\mathbf{W}^{-1}\mathbf{y}} \qquad \textit{under } \mathcal{H}_0$$

$$\mathbf{f}(\mathbf{y}; \Psi, \Phi) = \frac{1}{|\det(\mathbf{W})|}\mathbf{f}(\mathbf{u}; \Phi)\Big|_{\mathbf{u}=\mathbf{W}^{-1}(\mathbf{y}-\mu\mathbf{s})} \qquad \textit{under } \mathcal{H}_1$$

which in view of (8) and (11) reduces to

$$\mathbf{f}(\mathbf{y}; \Psi, \Phi) = \frac{1}{|\det(\mathbf{W})|}\prod_{n=1}^{N}\left(f(u_n; \Phi)\Big|_{u_n=\sum_{j=1}^{N}\omega_{nj}(\Psi)y_j}\right) \qquad \textit{under } \mathcal{H}_0 \tag{13a}$$

$$\mathbf{f}(\mathbf{y}; \Psi, \Phi) = \frac{1}{|\det(\mathbf{W})|}\prod_{n=1}^{N}\left(f(u_n; \Phi)\Big|_{u_n=\sum_{j=1}^{N}\omega_{nj}(\Psi)(y_j-\mu s_j)}\right) \textit{under } \mathcal{H}_1 \tag{13b}$$

Therefore the GLRT for this problem is to decide $\mathcal{H}_1$ if

$$\ell_G = \frac{\prod_{n=1}^{N} f\left(\sum_{j=1}^{N}\omega_{nj}(\hat{\hat{\Psi}})(y_j - \hat{\hat{\mu}}s_j); \hat{\hat{\Phi}}\right)}{\prod_{n=1}^{N} f\left(\sum_{j=1}^{N}\omega_{nj}(\hat{\Psi})y_j; \hat{\Phi}\right)} > \gamma \tag{14}$$

where hat's denote MLE under $\mathcal{H}_0$ and double hat's denote MLE under $\mathcal{H}_1$. It is assumed that the values of $|\det(\mathbf{W})|$ under H_0 and $\mathcal{H}_1$ are nearly the same. This assumption

simplifies the problem considerably. The threshold γ is adjusted to achieve a given probability of false alarm, as will be discussed in the next section.

Note that if Ψ is known so that $\mathbf{W}^{-1}\mathbf{y}$ can be computed, then (7) reduces to

$$\mathcal{H}_0 : \mathbf{W}^{-1}\mathbf{y} = \mathbf{u}$$

$$\mathcal{H}_1 : \mathbf{W}^{-1}\mathbf{y} = \mathbf{u} + \mu\mathbf{W}^{-1}\mathbf{s}$$

which is simply the problem of detecting the transformed signal $\mathbf{W}^{-1}\mathbf{s}$ of unknown amplitude μ in *i.i.d.* noise from the transformed observation vector $\mathbf{W}^{-1}\mathbf{y}$. The likelihood ratio corresponding to the GLRT for this problem is

$$\ell_G = \frac{\prod_{n=1}^{N} f\left(\sum_{j=1}^{N} \omega_{nj}(\Psi)(y_j - \hat{\mu}s_j); \hat{\hat{\Phi}}\right)}{\prod_{n=1}^{N} f\left(\sum_{j=1}^{N} \omega_{nj}(\Psi)y_j; \hat{\Phi}\right)}$$

The same statistic is used for the case of unknown Ψ by replacing it with its MLE under the respective hypotheses for numerator and denominator as per (14).

Another special case of (7) arises when the noise is white, *i.e.*, $\mathbf{W} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. (14) then reduces to [Kay 1985]

$$\ell_G = \frac{\prod_{n=1}^{N} f\left((y_n - \hat{\mu}s_n); \hat{\hat{\Phi}}\right)}{\prod_{n=1}^{N} f\left(y_n; \hat{\Phi}\right)} > \gamma$$

It was indicated earlier in this section that the linear model (7) is capable of representing the case of AR noise for large data records. The advantage of AR modeling of the noise as opposed to an ARMA or MA model is that it is easier to estimate the unknown parameters as required by the GLRT. This case is now examined in detail. The detection problem for AR noise is

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{y} = \mathbf{x} \\ \mathcal{H}_1 &: \mathbf{y} = \mathbf{x} + \mu\mathbf{s} \end{aligned} \tag{15}$$

with

$$\mathbf{x} = [x_1 \; x_2 \; \cdots \; x_N]^T$$

It is assumed that the sequence $\{x_1, x_2, \cdots, x_N\}$ is the output of a pth order all-pole filter excited by white driving noise or

$$x_n = -\sum_{j=1}^{p} a_j x_{n-j} + u_n, \qquad n = 1, 2, \cdots N$$

alternately,

$$u_n = x_n + \sum_{j=1}^{p} a_j x_{n-j} = \sum_{j=0}^{p} a_j x_{n-j}, \qquad n = 1, 2, \cdots N$$

assuming $a_0 = 1$. u_n can also be written as a function of $\mathbf{y}$ under either hypothesis

$$u_n = \sum_{j=0}^{p} a_j y_{n-j}, \qquad n = 1, 2, \cdots N \qquad \textit{under } \mathcal{H}_0 \qquad (16a)$$

$$= \sum_{j=0}^{p} a_j (y_{n-j} - \mu s_{n-j}), \qquad n = 1, 2, \cdots N \qquad \textit{under } \mathcal{H}_1 \qquad (16b)$$

Note that $u_1, u_2, \cdots u_p$ involves samples prior to y_1 which are outside the observation interval. These are assumed to be zero for simplicity. For large data records this assumption will not change the character of the GLRT. In the matrix form

$$\begin{pmatrix} u_1 \\ \vdots \\ u_p \\ u_{p+1} \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ \vdots & \ddots & & & & \\ a_p & \cdots & 1 & & & \\ 0 & a_p & \cdots & 1 & & \\ \vdots & \ddots & \ddots & & \ddots & \\ 0 & \cdots & 0 & a_p & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 - \mu s_1 \\ \vdots \\ y_p - \mu s_p \\ y_{p+1} - \mu s_{p+1} \\ \vdots \\ y_N - \mu s_N \end{pmatrix} \qquad (17)$$

under $\mathcal{H}_1$. The equation is the same under $\mathcal{H}_0$, except that $\mu = 0$. (17) is a special case of (11) with $\Psi = \mathbf{a} = [a_1 \; a_2 \; \cdots \; a_p]$ and $M = p$. $\mathbf{W}^{-1}$ is a lower triangular Toeplitz matrix given by

$$\omega_{ij}(\mathbf{a}) = \begin{cases} 0, & \text{if } i < j, \\ a_{i-j}, & \text{if } j \le i \le j + p, \\ 0, & \text{if } j + p < i. \end{cases}$$

To avoid having to assume that $\{y_{-(p-1)}, y_{-(p-2)}, \cdots, y_0\}$ are zero one can proceed as follows. Considering only the last $(N-p)$ equations of (16) which expressed in the matrix form are

$$\begin{pmatrix} u_{p+1} \\ \vdots \\ u_{2p} \\ u_{2p+1} \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ \vdots & \ddots & & & & & \\ a_p & \dots & 1 & & & & \\ 0 & a_p & \dots & 1 & & & \\ \vdots & \ddots & \ddots & & \ddots & & \\ 0 & \dots & 0 & a_p & \dots & 1 \end{pmatrix} \begin{pmatrix} y_{p+1} - \mu s_{p+1} \\ \vdots \\ y_{2p} - \mu s_{2p} \\ y_{2p+1} - \mu s_{2p+1} \\ \vdots \\ y_N - \mu s_N \end{pmatrix}$$

$$+ \begin{pmatrix} \sum_{j=1}^{p} a_{p-j+1}(y_j - \mu s_j) \\ \sum_{j=2}^{p} a_{p-j+2}(y_j - \mu s_j) \\ \vdots \\ a_p(y_p - \mu s_p) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{18}$$

under $\mathcal{H}_1$. Substitution of $\mu = 0$ in (18) gives the corresponding equation for $\mathcal{H}_0$. (18) is also a special case of (11b) except that only the last $(N-p)$ of the N scalar equations implied by (11b) are used. Since the added vector causes a departure from the general model, the previous results can not be used. To determine the GLRT first consider the conditional likelihood function. In this case the conditional likelihood of $y_{p+1}, y_{p+2}, \cdots, y_N$ given $y_1, y_2, \cdots, y_p$ is

$$\mathbf{f}(y_{p+1}, y_{p+2}, \cdots, y_N | y_1, y_2, \cdots, y_p)$$

$$= \prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} a_j y_{n-j}; \Phi\right) \qquad \text{under } \mathcal{H}_0 \tag{19a}$$

$$= \prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} a_j (y_{n-j} - \mu s_{n-j}); \Phi\right) \qquad \text{under } \mathcal{H}_1 \tag{19b}$$

The likelihood ratio is given by

$$\ell_G = \frac{\mathrm{f}(y_{p+1}, y_{p+2}, \cdots, y_N | y_1, y_2, \cdots, y_p; \hat{\hat{\Theta}}_r, \hat{\hat{\Theta}}_s)\ \mathrm{f}(y_1, y_2, \cdots, y_p; \hat{\hat{\Theta}}_r, \hat{\hat{\Theta}}_s)}{\mathrm{f}(y_{p+1}, y_{p+2}, \cdots, y_N | y_1, y_2, \cdots, y_p; 0, \hat{\Theta}_s)\ \mathrm{f}(y_1, y_2, \cdots, y_p; 0, \hat{\Theta}_s)}$$

$$= \left[\frac{\prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} \hat{\hat{a}}_j (y_{n-j} - \hat{\mu} s_{n-j}); \hat{\hat{\Phi}} \right)}{\prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} \hat{a}_j y_{n-j}; \hat{\Phi} \right)} \right] \left[\frac{\mathrm{f}(y_1, y_2, \cdots, y_p; \hat{\mu}, \hat{\hat{\mathrm{a}}}, \hat{\hat{\Phi}}}{\mathrm{f}(y_1, y_2, \cdots, y_p; 0, \hat{\mathrm{a}}, \hat{\Phi}} \right]$$

where $\hat{\hat{a}}_0$ and $\hat{a}_0$ are defined to be unity. The second term is dropped for ease of computation. A heuristic justification for ignoring the second term is that when N is large, its contribution to ℓ_G will be negligible. The closer the poles of the AR model to the unit circle, larger is the requirement for N [Box and Jenkins 1970], [Kay 1981]. With this simplification, the GLRT decides $\mathcal{H}_1$ if

$$\ell_G = \left[\frac{\prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} \hat{\hat{a}}_j (y_{n-j} - \hat{\mu} s_{n-j}); \hat{\hat{\Phi}} \right)}{\prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} \hat{a}_j y_{n-j}; \hat{\Phi} \right)} \right] > \gamma \tag{20}$$

A comparison of (14) and (20) shows that the latter uses fewer terms in the product. However both formulations are clearly asymptotically eqivalent. Figure 1 is a block diagram to generate $2 \ln \ell_G$ from the data. The reason for computing $2 \ln \ell_G$ instead of ℓ_G will be clear from the discussion in the next section. The block diagram is very much similar to that obtained by [Kay 1983] for the detection of a *completely known* signal in unknown colored *Gaussian* noise. In the Gaussian case $\ln f$ is a simple quadratic, while for the general non-Gaussian case it will be highly non-linear. Figure 1 also uses an *estimator* for μ which was assumed to be known in [Kay 1983].

IV. Asymptotic Performance of the GLRT Detector

Asymptotic distributions of $2 \ln \ell_G$ under $\mathcal{H}_0$ and $\mathcal{H}_1$ are given by ($3a$) and ($3b$) respectively. In this case $\Theta_r = \mu$, $\Theta_s = [\Psi^T \ \Phi^T]^T$ for the general linear model and

$\Theta_s = [\mathrm{a}^T \ \Phi^T]^T$ for the AR case. Hence the noncentrality parameter is

$$\lambda = \mu^2 \left[\mathbf{I}_{\mu\mu}(0, \Theta_s) - \mathbf{I}_{\mu\Theta_s}(0, \Theta_s) \mathbf{I}^{-1}_{\Theta_s\Theta_s}(0, \Theta_s) \mathbf{I}^T_{\mu\Theta_s}(0, \Theta_s) \right] \tag{21}$$

The probability of false alarm is

$$P_{FA} = \mathcal{P}\{2 \ln \ell_G > \gamma' | \mathcal{H}_0\} \tag{22}$$

where $\gamma' = 2 \ln \gamma$. The probability of detection is

$$P_D = \mathcal{P}\{2 \ln \ell_G > \gamma' | \mathcal{H}_1\} \tag{23}$$

Both the probabilities can be calculated from the tables of noncentral and central chi-squared distributions, respectively. In practice, γ' can be set to produce a given false alarm rate and P_D can be calculated from (23) accordingly.

As indicated before, there is no UMP test for the detection problem considered in this paper. Therefore there is no upper bound to which the performance of any detector may be compared. However the performance of the GLRT is better appreciated when compared to that of a *clairvoyant* GLRT. A clairvoyant GLRT is one which uses *perfect knowledge of the nuisance parameters* Θ_s. The likelihood ratio in this case is

$$\ell_{GC} = \frac{\mathbf{f}(\mathbf{y}; \hat{\hat{\Theta}}_r, \Theta_s)}{\mathbf{f}(\mathbf{y}; \mathbf{0}, \Theta_s)}$$

which in view of (13) is

$$\ell_{GC} = \frac{\displaystyle\prod_{n=1}^{N} f\left(\sum_{j=1}^{N} \omega_{nj}(\Psi)(y_j - \hat{\mu} s_j); \Phi\right)}{\displaystyle\prod_{n=1}^{N} f\left(\sum_{j=1}^{N} \omega_{nj}(\Psi) y_j; \Phi\right)}$$

where Ψ and Φ are assumed to be known. Asymptotically, $2 \ln \ell_{GC}$ is distributed as

$$2 \ln \ell_{GC} \sim \chi^2_r \qquad \text{under } \mathcal{H}_0 \tag{24a}$$

$$2 \ln \ell_{GC} \sim \chi'^2(r, \lambda_c) \qquad \text{under } \mathcal{H}_1 \tag{24b}$$

where

$$\lambda_c = \Theta_r^T \mathbf{I}_{\Theta_r\Theta_r}(0, \Theta_s)\Theta_r$$

For the problem considered here, $r = 1$ and $\Theta_r = \mu$. Hence

$$\lambda_c = \mu^2 I_{\mu\mu}(0, \Theta_s) \tag{25}$$

Comparing λ and λ_c as given by (21) and (25), respectively, it is apparent that λ is equal to λ_c less an additional term. Assuming that $I^{-1}_{\Theta_s\Theta_s}$ is positive semidefinite,

$$\lambda_c \geq \lambda$$

From the theory of noncentral chi-square distribution it can be shown that P_D as given by (23) is a monotonic function of the noncentrality parameter, which implies that P_D for the clairvoyant GLRT is greater than or equal to that for the GLRT [Sengupta 1986]. Therefore the clairvoyant GLRT detector, although impractical in this case, provides an upper bound on the performance of the GLRT detector. In order that the upper bound be *achieved*, λ should be *equal to* λ_c. This will occur if

$$\mathbf{I}_{\mu\Theta_s}(0, \Theta_s) = \mathbf{0} \tag{26}$$

Appendices A and B show that this is indeed the case for the general linear model of (7) and the AR noise model of (15), respectively, if as assumed f is a symmetric PDF. Therefore *the asymptotic performance of the GLRT is equivalent to the performance of the clairvoyant GLRT* for detection in presence on non-Gaussian noise modeled as in (7) *or* (15). This implies that *one can do as well in detecting a signal of unknown amplitude as if the unknown noise parameters were known.*

V. General Conclusions about the Performance of the GLRT

A key to the asymptotic performance of the GLRT detector is the noncentrality parameter λ, which is found to be equivalent to λ_c. It would be interesting to examine

how λ depends on the statistical properties of the noise. First consider

$$
\begin{aligned}
I_{\mu\mu}(\mu,\Theta_s) &= E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial\mu}\right)^2\right] \\
&= E\left[\left[\frac{\partial \ln\left(\prod_{n=1}^{N} f\left(\sum_{j=1}^{N}\omega_{nj}(y_j-\mu s_j);\Phi\right)\right)}{\partial\mu}\right]^2\right] \\
&= E\left[\left[\frac{\partial \ln\left(\prod_{n=1}^{N} f(u_n;\Phi)\right)}{\partial\mu}\right]^2\right] \qquad \text{using } (11b) \text{ and } (12) \\
&= \sum_{n=1}^{N} E\left[\left[\frac{\partial}{\partial\mu}\ln f(u_n;\Phi)\right]^2\right] \qquad (27)
\end{aligned}
$$

The last step follows from the facts that u_n's are *i.i.d.* and that the cross-terms are zero, since

$$
E\left[\frac{\partial}{\partial\mu}\ln f(u_n;\Phi)\right] = \frac{\partial}{\partial\mu}E[\ln f(u_n;\Phi)] = 0
$$

under certain regularity assumptions on f [Bickel and Doksum 1977]. Writing (11*b*) explicitly as

$$
u_n = \sum_{j=1}^{N}\omega_{nj}(y_j-\mu s_j) \qquad (28)
$$

it follows that

$$
\frac{\partial u_n}{\partial\mu} = -\sum_{j=1}^{N}\omega_{nj}s_j \qquad (29)
$$

Hence (27) can be rewritten as

$$
\begin{aligned}
I_{\mu\mu}(\mu,\Theta_s) &= \sum_{n=1}^{N} E\left[\left[\left(\frac{\partial}{\partial u_n}\ln f(u_n;\Phi)\right)\left(\frac{\partial u_n}{\partial\mu}\right)\right]^2\right] \\
&= \sum_{n=1}^{N} E\left[\left(-\sum_{j=1}^{N}\omega_{nj}s_j\right)^2\left(\frac{\partial \ln f}{\partial u_n}\right)^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \left(-\sum_{j=1}^{N} \omega_{nj} s_j \right)^2 E\left[\left(\frac{\partial \ln f}{\partial u_n} \right)^2 \right] \\
&= \sum_{n=1}^{N} \left(-\sum_{j=1}^{N} \omega_{nj} s_j \right)^2 I_f(\Phi) \\
&= \left[\frac{1}{\sigma^2} \sum_{n=1}^{N} \left(-\sum_{j=1}^{N} \omega_{nj} s_j \right)^2 \right] \sigma^2 I_f(\Phi) \qquad (30)
\end{aligned}
$$

where σ^2 is the variance of u_n and

$$
I_f(\Phi) = E\left[\left(\frac{\partial \ln f}{\partial u_n} \right)^2 \right]
$$

does not depend on μ or u_n. The expectation is with respect to u_n only since u_n's are *i.i.d.* It follows from (25) that

$$
\begin{aligned}
\lambda = \lambda_c &= \left[\frac{\mu^2}{\sigma^2} \sum_{n=1}^{N} \left(-\sum_{j=1}^{N} \omega_{nj} s_j \right)^2 \right] \sigma^2 I_f(\Phi) \\
&= \frac{\mu^2}{\sigma^2} (\mathbf{W}^{-1}\mathbf{s})^T (\mathbf{W}^{-1}\mathbf{s}) \sigma^2 I_f(\Phi) \\
&= \frac{\mu^2}{\sigma^2} \mathbf{s}^T (\mathbf{W}^{-1^T} \mathbf{W}^{-1}) \mathbf{s} \sigma^2 I_f(\Phi) \\
&= (\mu \mathbf{s}^T)(\sigma^2 \mathbf{W}\mathbf{W}^T)^{-1} (\mu \mathbf{s})^T \sigma^2 I_f(\Phi) \\
&= \mathbf{s}_0^T \mathbf{R}^{-1} \mathbf{s}_0 [\sigma^2 I_f(\Phi)] \qquad (31)
\end{aligned}
$$

where $\mathbf{R} = \sigma^2 \mathbf{W}\mathbf{W}^T$ is the $N \times N$ autocorrelation matrix of the colored noise and $\mathbf{s}_0 = \mu \mathbf{s}$ is the signal vector including the amplitude. $\mathbf{s}_0^T \mathbf{R}^{-1} \mathbf{s}_0$ is the signal to noise ratio (SNR) at the output of a prewhitener followed by a matched filter (or correlator), both built with perfect knowledge of the filter parameters (Ψ). To be more precise, if the data is passed through an ideal whitener (a filter which will completely whiten the *noise*) and correlated (multiplied term-by-term and summed) with the output of a similar filter through which only the *known signal* is passed then $\mathbf{s}_0^T \mathbf{R}^{-1} \mathbf{s}_0$ is the

squared ratio of the contributions from the signal and noise parts of the data. In the case of AR noise, a similar derivation using (16*b*) (instead of (28)) gives

$$I_{\mu\mu}(\mu, \Theta_s) = \left[\frac{1}{\sigma^2} \sum_{n=p+1}^{N} \left(-\sum_{j=0}^{p} a_j s_{n-j} \right)^2 \right] \sigma^2 I_f(\Phi) \tag{32}$$

and

$$\begin{aligned} \lambda &= \left[\frac{\mu^2}{\sigma^2} \sum_{n=p+1}^{N} \left(-\sum_{j=0}^{p} a_j s_{n-j} \right)^2 \right] \sigma^2 I_f(\Phi) \\ &= \frac{\mu^2}{\sigma^2} (\mathbf{As})^T (\mathbf{As}) \sigma^2 I_f(\Phi) \end{aligned} \tag{33}$$

where $\mathbf{A}$ is the $(N-p) \times N$ Toeplitz matrix

$$\mathbf{A} = \begin{pmatrix} a_p & \dots & a_1 & 1 & 0 & \dots & 0 \\ 0 & a_p & \dots & a_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_p & \dots & a_1 & 1 \end{pmatrix}$$

Therefore

$$\lambda = \mathbf{s}_0^T \mathbf{R}^{-1} \mathbf{s}_0 [\sigma^2 I_f(\Phi)] \tag{34}$$

$\mathbf{s}_0 = \mu \mathbf{s}$ as before and $\mathbf{R} = \sigma^2 (\mathbf{A}^T \mathbf{A})^{-1}$ is approximately the covariance matrix of the noise. Clearly, λ is proportional to the SNR at the output of a prewhitener-correlator (using true value of $\mathbf{a}$) in the AR case also.

Having established similar results in the cases of AR noise and the general linear model, an attempt is now made to examine them. The AR noise model is chosen for this purpose because of its intuitive frequency-domain interpretation. Figure 2 is a block diagram representing (33). It shows that λ can be obtained by inverse filtering the signal and summing the squares of the output of the filter. If the signal has most of its power at the frequency where the inverse filter has a zero, the output power and hence λ will be small leading to a small probability of detection. In other words, *it*

is difficult to detect the signal if the peaks of the signal spectrum coincides with the peaks of the noise PSD. This makes perfect intuitive sense. On the other hand it is possible to maximize λ by choosing a suitable signal $\mathbf{s}$ for a given noise background. This can be done by constraining the signal energy to be constant and maximizing the SNR at the output of a prewhitener-correlator over all possible signal shapes. Writing the Toeplitz matrix $\mathbf{R}$ in terms of its orthonormal eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_N\}$ and eigenvalues $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_N\}$

$$\mathbf{R} = \sum_{j=1}^{N} \Lambda_j \mathbf{v}_j \mathbf{v}_j^T \tag{35}$$

it follows that

$$\mathbf{R}^{-1} = \sum_{j=1}^{N} \frac{1}{\Lambda_j} \mathbf{v}_j \mathbf{v}_j^T$$

Hence

$$\begin{aligned} \mathbf{s}_0^T \mathbf{R}^{-1} \mathbf{s}_0 &= \sum_{j=1}^{N} \frac{1}{\Lambda_j} (\mathbf{s}_0^T \mathbf{v}_j)^2 \\ &= \sum_{j=1}^{N} \frac{1}{\Lambda_j} \varsigma_j^2 \end{aligned} \tag{36}$$

where $\varsigma_j = \mathbf{s}_0^T \mathbf{v}_j$ is the component of the signal $\mathbf{s}_0$ along the eigenvector $\mathbf{v}_j$. The condition of constant signal energy can be written as

$$\mathbf{s}_0^T \mathbf{s}_0 = P_s \tag{37}$$

Since the eigenvectors are orthonormal

$$\sum_{j=1}^{N} \varsigma_j^2 = \sum_{j=1}^{N} \mathbf{s}_0^T \mathbf{v}_j \mathbf{v}_j^T \mathbf{s}_0 = \mathbf{s}_0^T \left[\sum_{j=1}^{N} \mathbf{v}_j \mathbf{v}_j^T \right] \mathbf{s}_0 = \mathbf{s}_0^T \mathbf{s}_0 = P_s \tag{38}$$

The SNR given by the weighted sum (36) has to be maximized subject to constraint that the unweighted sum of the squares is fixed at P_s as in (38). In general, $\mathbf{R}$ will be positive definite and all the eigenvalues will be positive. If there exists a minimum

eigenvalue Λ_k, then the SNR is maximized by choosing $\varsigma_k = \sqrt{P_s}$ and $\varsigma_j = 0$ for $j \neq k$, *i.e.*, by choosing $\mathbf{s}_0$ to be proportional to $\mathbf{v}_k$. Since the probability of detection given by (23) is a monotonic function of λ which is proportional to the SNR at the output of a prewhitener-correlator, the above choice of the signal shape for a given signal energy also maximizes the probability of detection. If one of the eigenvalues Λ_k is zero, then it is possible to chose the signal in such a way that there is no component of noise along the signal vector and therefore the SNR is infinite giving rise to *singular detection.* Therefore *the probability of detection is maximized by choosing the signal in the direction of the smallest noise component.* This is the discrete time equivalent of a well-known result for the continuous case [Van Trees 1968]. An interesting special case occurs when $N \to \infty$ such that the eigenvectors become

$$\mathbf{v}_j \to \frac{1}{\sqrt{N}}\left(1 \; e^{j2\pi f_j} \; \ldots \; e^{j2\pi(N-1)f_j}\right)$$

Hence the optimum signal is a *sinusoid* in the direction of the eigenvector associated with the minimum eigenvalue. For very large data records the eigenvalue Λ_j corresponding to the eigenvector $\mathbf{v}_j$ approaches the value of PSD at the frequency f_j. Hence the signal easiest to detect would be a sinusoid at the frequency at which the noise PSD has a minimum. This is also apparent from the frequency domain equivalent of (36) (using Perseval's theorem)

$$\mathbf{s}_0^T \mathbf{R}^{-1} \mathbf{s}_0 = \int_{-0.5}^{0.5} \frac{|S_0(f)|^2}{P_{uu}(f)} df$$

where $S_0(f)$ is the signal spectrum (Fourier transform of $\mathbf{s}_0$) and P_{uu} is the noise PSD. With the constraint that

$$\int_{-0.5}^{0.5} |S_0(f)|^2 df = 1$$

which is equivalent to (37), the integral is maximized if the numerator is large only where the denominator is small or zero. This result has a nice intuitive justification.

However if the filter parameters are completely unknown the above result can not be used to select a suitable signal.

The next issue of interest is the effect of the noise PDF on the detection performance. A reasonable basis of comparison should be formed for this purpose. Therefore the Gaussian and non-Gaussian noise processes are assumed to have the same PSD, *i.e.*, the same spectral *shape* and *power* and detection of the same signal is considered. Consequently, the comparison is done on the basis of a fixed signal to noise ratio (or $\mathbf{s}_0^T \mathbf{R}^{-1} \mathbf{s}_0$). Under these assumptions, the probability of detection is larger for that noise PDF which has a larger value of $\sigma^2 \mathcal{I}_f$. In other words, given two noise backgrounds with the same PSD but different underlying noise PDF's, in order to achieve the same probability of detection, more SNR is required for that background for which $\sigma^2 \mathcal{I}_f$ is smaller. It is known that among all symmetric and integrable PDF's, the Gaussian PDF is the only one for which $\sigma^2 \mathcal{I}_f$ attains its minimum value of unity [Sengupta and Kay 1986]. Therefore for a given noise PSD, *it is easier to detect a signal known except for amplitude in non-Gaussian noise than in Gaussian noise.* From (34) it follows that in order to have the same noncentrality parameters in the non-Gaussian and Gaussian cases

$$\mathcal{SNR}_{NG}(\sigma^2 \mathcal{I}_f) = \mathcal{SNR}_G$$

where $\mathcal{SNR}_{NG}$ and $\mathcal{SNR}_G$ are the SNR's required in non-Gaussian and Gaussian noises, respectively, in order to achieve a given probability of detection (*i.e.*, a given λ). The above equation can also be written as

$$10 \log_{10} \frac{\mathcal{SNR}_G}{\mathcal{SNR}_{NG}} = 10 \log_{10}(\sigma^2 \mathcal{I}_f) \tag{39}$$

Therefore $10 \log_{10}(\sigma^2 \mathcal{I}_f)$ is a *measure of the SNR bonus in dB for a non-Gaussian distribution.* The result also holds for the special case of white noise when (33) becomes

$$\lambda = \left[\frac{\mu^2}{\sigma^2} \sum_{n=1}^{N} s_n^2 \right] \sigma^2 \mathcal{I}_f(\Phi) \tag{40}$$

The quantity $\sigma^2 I_f$ is now shown to be independent of scaling. If the random variable u has a PDF $f(u)$ then the normalized random variable $\dot{u} = u/\sigma$ has the PDF

$$g(\dot{u}) = \sigma f(u)$$

hence

$$\begin{aligned}
\sigma^2 I_f &= \sigma^2 \int_{-\infty}^{\infty} \frac{[f'(u)]^2}{f(u)} du \\
&= \sigma^2 \int_{-\infty}^{\infty} \frac{\left[\frac{1}{\sigma} g'(\frac{u}{\sigma}) \frac{1}{\sigma}\right]^2}{\frac{1}{\sigma} g(\frac{u}{\sigma})} du \\
&= \sigma^2 \int_{-\infty}^{\infty} \frac{1}{\sigma^4} \frac{(g'(\dot{u}))^2}{\frac{1}{\sigma} g(\dot{u})} \sigma d\dot{u} \\
&= \int_{-\infty}^{\infty} \frac{(g'(\dot{u}))^2}{g(\dot{u})} d\dot{u}
\end{aligned}$$

Hence $\sigma^2 I_f$ depends only on the *shape* of the PDF and is unaffected by scaling. Therefore the SNR bonus quantified by (39) is the same for any value of the noise power as long as the powers of the non-Gaussian and Gaussian processes are the same.

It is interesting to note that the same quantity represents the amount of departure from Gaussianity for the problem of estimating AR filter parameters of a non-Gaussian AR process. The CR bounds for these parameters are found to be less in the case of a non-Gaussian PDF than the corresponding bounds in the Gaussian case by a factor of $\sigma^2 I_f$ [Sengupta and Kay 1986].

As an illustration of the improvement made by the proposed detector over the Gaussian detector, consider the *mixed-Gaussian* noise PDF

$$f(u) = \frac{1-\epsilon}{\sqrt{2\pi\sigma_B^2}} e^{-\frac{u^2}{2\sigma_B^2}} + \frac{\epsilon}{\sqrt{2\pi\sigma_I^2}} e^{-\frac{u^2}{2\sigma_I^2}}$$

The first term on the right hand side is referred to as the *background* component with variance σ_B^2 and the second term is called the *interference* component with variance σ_I^2. ϵ is called the mixture parameter and is regarded as a measure of the degree of

contamination of the background Gaussian process by the interference process. The model is useful in representing a nominally Gaussian noise background characterized by the presence of sharp spikes or impulses [Sengupta and Kay 1986]. Assuming $\sigma_B^2 = 1$ and $\sigma_I^2 = 1000$, Figure 3 plots the SNR bonus given by (39) vs. ϵ (in this case $\Phi = \epsilon$). It shows how much improvement can be expected over the Gaussian case in terms of SNR while detecting a signal known except for amplitude in colored noise. The comparison is made, as indicated before on the basis of the *same* PSD in the Gaussian and mixed-Gaussian cases. It should be mentioned, however, that introduction of impulses in an otherwise Gaussian environment *does not improve* the probability of detection, which is expected intuitively. This is because of the fact that introduction of impulses also increases the noise power by a considerable amount. This *increase* in noise power is *alleviated* by employing a non-Gaussian detector. As an example, for $\epsilon = 0.1$, the overall noise variance is approximately $100\sigma_B^2$ (as compared to σ_B^2 before the introduction of impulses), *i.e.*, the noise power increases by 20 dB. It can be observed from Figure 3 that the SNR bonus is also approximately 20 dB for $\epsilon = 0.1$. Therefore the mixed-Gaussian detector does not suffer from a loss of performance unlike the Gaussian detector whose threshold of detection is expected to go down considerably with the introduction of impulses.

VI. Summary

The GLRT for the detection of a signal known except for amplitude in unknown colored non-Gaussian noise was derived in section III through parametric modeling of the noise PDF and covariance matrix. The popular time series models such as AR, MA and ARMA for the noise are asymptotically special cases of the proposed linear model for large data records. The GLRT was found to achieve the performance of a clairvoyant GLRT asymptotically, *i.e.*, knowledge of the nuisance parameters is not required to attain an upper bound in performance. The effects of the signal spectrum

and the noise PSD on the detection performance was discussed. It was observed that it is difficult to detect a signal whose spectrum matches the noise PSD. If, however, most of the signal is along a direction of low noise component, it is very easy to detect. The asymptotic performance of the GLRT for Gaussian and non-Gaussian noise models were compared. It was concluded that detection in non-Gaussian noise is easier than detection in Gaussian noise for the same noise PSD. The improvement in performance of the GLRT in a non-Gaussian noise background over the Gaussian case is easily quantified in terms of the SNR 'bonus' as a function of the noise PDF parameters.

In order to implement the GLRT described in section III one needs to find the MLE's of the unknown parameters under each hypothesis. Some work along this line has been done for the case of AR noise [Sengupta and Kay 1986, 2], using reasonable approximations to reduce computation. This work is approriate for estimation under the null hypothesis ($\mathcal{H}_1$) and extension to the case of alternative hypothesis ($\mathcal{H}_1$) is not straightforward except for the special case of a d.c. signal ($s_j = 1, \quad j = 1, 2, \cdots, N$). Evaluating the joint MLE of the mean or the location parameter (μ) and the AR filter parameters may be particularly difficult for most non-Gaussian processes. Computationally efficient approximations to the GLRT, such as the Rao test and the Wald test [Rao 1973] can be used for this purpose. Estimation of the mean and the other parameters under $\mathcal{H}_1$ can thus be avoided for small signal amplitudes [Sengupta 1986]. This problem will be addressed in a future paper.

References

[1] H.L. Van Trees, *Detection, Estimation, and Modulation Theory*, Chapter 4, New York: John Wiley, 1968.

[2] A.D. Whalen, *Detection of Signals in Noise*, Chapter 9, New York: Academic, 1971.

[3] D.E. Bowyer *et al*, "Adaptive Clutter Filtering using Autoregressive Spectral Estimation", IEEE Trans. on Aerosp. Electron. Syst., pp. 538–546, July 1979.

[4] S.M. Kay, "Asymptotically Optimal Detection in Unknown Colored Noise via Autoregressive Modeling", IEEE Trans. on Acoustics, Speech and Signal Processing, pp. 927–940, Vol. ASSP-31, Aug. 1983.

[5] W.C. Knight, R.G. Pridham and S.M. Kay, "Digital Signal Processing for Sonar", Proc. of the IEEE, pp. 1451–1506, Nov. 1981.

[6] S.C. Lee, L.W. Nolte and C.P. Hatsell, "A Generalized Likelihood Ratio Formula: Arbitrary Noise Statistics for Doubly Composite Hypotheses", IEEE Trans. on Info. Theory, pp. 637–639, Vol. IT-23, Sept. 1977.

[7] S.A. Kassam and H.V. Poor, "Robust Techniques for Signal Processing", Proc. of the IEEE, pp. 433–481, Vol. 73, Mar. 1985.

[8] A.B. Martinez and J.B. Thomas, "Non-Gaussian Multivariate Noise Models for Signal Detection", ONR report #6, Sept. 1982.

[9] S.V. Czarnecki and J.B. Thomas, "Nearly Optimal Detection of Signals in Non-Gaussian Noise", ONR report #14, Feb. 1984.

[10] S.M. Kay, "Asymptotically Optimal Detection in Incompletely Characterized Non-Gaussian Noise", Submitted to IEEE Trans. on Acoustics, Speech and Signal Processing, 1985.

[11] Sir M. Kendall and A. Stuart, *The Advanced Theory of Statistics Vol. II*, Chapters 18–19, New York: MacMillan Publishing, 1979.

[12] E.L. Lehmann, *Testing Statistical Hypotheses*, New York: John Wiley, 1959.

[13] D.R. Cox and D.V. Hinkley, *Theoretical Stastics*, Chapter 4, London: Chapman and Hall, 1974.

[14] G.E.P. Box and G.J. Jenkins, *Time Series Analysis: Forecasting and Control*, Chapter 7, San Francisco: Holden-Day, 1970.

[15] S.M. Kay, "More Accurate Autoregressive Parameter and Spectral Estimates

for Short Data Records", presented at the ASSP workshop on Spectral Estimation, Hamilton, Onterio, Canada, Aug. 17–18, 1981.

[16] D. Sengupta, "Estimation and Detection for Non-Gaussian Processes using Autoregressive and Other Models", M.S. Thesis, Dept. of Electrical Engineering, Univ. of Rhode Island, 1986.

[17] P.J. Bickel and K.A. Doksum, *Mathematical Statistics: Basic Ideas and Selected Topics*, Chapter 4, San Francisco: Holden-Day, 1977.

[18] D. Sengupta and S.M. Kay, "Efficient Estimation of Parameters for Non-Gaussian Autoregressive Processes", submitted for review to IEEE Trans. on Acoustics, Speech and Signal Processing.

[19] S.M. Kay and D. Sengupta, "Simple and Efficient Estimation of Parameters of Non-Gaussian Autoregressive Processes", submitted for review to IEEE Trans. on Acoustics, Speech and Signal Processing.

[20] C.R. Rao, *Linear Statistical Inference and its Applications*, Chapter 6, New York: John Wiley, 1973.

APPENDIX A

Asymptotic Optimality of the GLRT for a General Linear Model of the Noise

Assuming that f is an even distribution and Θ_s is as given in (10), it will now be shown that (26) holds for the detection problem defined in (7). It suffices to prove that

$$I_{\mu\Psi}(\mu,\Psi,\Phi) = E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\delta \Psi}\right)^T\right] = 0 \qquad (A.1)$$

and

$$I_{\mu\Phi}(\mu,\Psi,\Phi) = E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\delta \Phi}\right)^T\right] = 0 \qquad (A.2)$$

To prove $(A.1)$ it is observed that

$$E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\delta \psi_j}\right)\right] = E\left[\left(\frac{\partial \ln \mathrm{f}}{\delta \mu}\right)\sum_{i=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial \ln \mathrm{f}}{\partial \omega_{ik}}\right)\left(\frac{\partial \omega_{ik}}{\partial \psi_j}\right)\right]$$
$$= \sum_{i=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial \omega_{ik}}{\partial \psi_j}\right) E\left[\left(\frac{\partial \ln \mathrm{f}}{\delta \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\partial \omega_{ik}}\right)\right] \qquad (A.3)$$

ω_{ik} is written without its argument (see (12)) to make the notation easier.

$$E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\partial \omega_{ik}}\right)\right]$$

$$= E\left[\frac{\partial \ln\left(\prod_{n=1}^{N} f\left(\sum_{j=1}^{N}\omega_{nj}(y_j - \mu s_j);\Phi\right)\right)}{\partial \mu}\;\frac{\partial \ln\left(\prod_{n=1}^{N} f\left(\sum_{j=1}^{N}\omega_{nj}(y_j - \mu s_j);\Phi\right)\right)}{\partial \omega_{ik}}\right]$$

u_n can be used as the argument of f (see (28)) to simplify the equation.

$$E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\partial \omega_{ik}}\right)\right] = E\left[\frac{\partial \ln\left(\prod_{n=1}^{N} f(u_n;\Phi)\right)}{\partial \mu}\;\frac{\partial \ln\left(\prod_{n=1}^{N} f(u_n;\Phi)\right)}{\partial \omega_{ik}}\right]$$
$$= \sum_{n=1}^{N}\sum_{m=1}^{N} E\left[\left(\frac{\partial}{\partial \mu}\ln f(u_n;\Phi)\right)\left(\frac{\partial}{\partial \omega_{ik}}\ln f(u_m;\Phi)\right)\right]$$
$$= \sum_{n=1}^{N} E\left[\left(\frac{\partial}{\partial \mu}\ln f(u_n;\Phi)\right)\left(\frac{\partial}{\partial \omega_{ik}}\ln f(u_n;\Phi)\right)\right] \qquad (A.4)$$

All the cross-terms are zero because u_n's are $i.i.d.$ and

$$E\left[\frac{\partial}{\partial \mu}\ln f(u_n;\Phi)\right] = \frac{\partial}{\partial \mu}E[\ln f(u_n;\Phi)] = 0 \qquad (A.5)$$

under certain regularity assumptions on f [Bickel and Doksum 1977]. The derivatives $w.r.t.$ μ and ω_{ik} can be written in terms of the derivative $w.r.t.$ u_n. Note from (28) that

$$\frac{\partial u_n}{\partial \omega_{ik}} = \begin{cases} y_k - \mu s_k & \text{if } n = i \\ 0 & \text{if } n \neq i \end{cases} \qquad (A.6)$$

From $(A.6)$ and (29) it follows that $(A.4)$ can be rewritten as

$$E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\partial \omega_{ik}}\right)\right] = \sum_{n=1}^{N} E\left[\left(\frac{\partial \ln f}{\partial u_n}\right)\left(\frac{\partial u_n}{\partial \mu}\right)\left(\frac{\partial \ln f}{\partial u_n}\right)\left(\frac{\partial u_n}{\partial \omega_{ik}}\right)\right]$$

$$= \left(-\sum_{j=1}^{N}\omega_{ij}s_j\right) E \underbrace{\left[\underbrace{\left[\frac{\partial}{\partial u_i}\ln f(u_i;\Phi)\right]^2}_{even}\underbrace{\left[y_k - \mu s_k\right]}_{odd}\right]}_{odd}$$

$(y_k - \mu s_k)$ is a linear function of $\mathbf{u}$, as observed from (7). The PDF f is even and expectation is taken on a function which is odd over each u_n. Therefore the expectation must be zero.

$$E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\partial \omega_{ik}}\right)\right]$$

$$= \left(-\sum_{j=1}^{N}\omega_{ij}s_j\right) E\left[\left[\frac{\partial}{\partial u_i}\ln f(u_i;\Phi)\right]^2\left(\sum_{j=1}^{N}\omega_{kj}u_j\right)\right]$$

$$= \left(-\sum_{j=1}^{N}\omega_{ij}s_j\right)\sum_{j=1}^{N}\omega_{kj}E\left[\left[\frac{\partial}{\partial u_i}\ln f(u_i;\Phi)\right]^2 u_j\right]$$

$$= \left(-\sum_{j=1}^{N}\omega_{ij}s_j\right)\sum_{j=1}^{N}\omega_{kj}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial u_i}\ln f(u_i;\Phi)\right]^2 u_j\left(\prod_{m=1}^{N} f(u_m;\Phi)\right) du_1 du_2\cdots du_N$$

$$= \left(-\sum_{j=1}^{N}\omega_{ij}s_j\right)\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial u_i}\ln f(u_i;\Phi)\right]^2 f(u_i;\Phi)du_i \sum_{\substack{j=1\\ j\neq i}}^{N}\omega_{kj}\underbrace{\int_{-\infty}^{\infty} u_j f(u_j;\Phi)du_j}_{0}$$

$$+\,\omega_{ki}\underbrace{\int_{-\infty}^{\infty}\underbrace{\left[\frac{\partial}{\partial u_i}\ln f(u_i;\Phi)\right]^2}_{even}\underbrace{u_i f(u_i;\Phi)}_{odd}du_i}_{0}$$

$$= 0$$

This is true for each i and k, so that

$$E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\partial \omega_{ik}}\right)\right] = 0 \qquad i,k = 1,2,\cdots,N \tag{A.7}$$

$(A.1)$ follows directly from $(A.3)$ and $(A.7)$.

$(A.2)$ can be proved in a similar way. Consider

$$E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathbf{f}}{\partial \phi_i}\right)\right]$$

$$= E\left[\frac{\partial \ln\left(\prod_{n=1}^{N} f\left(\sum_{j=1}^{N}\omega_{nj}(y_j - \mu s_j);\Phi\right)\right)}{\partial \mu} \frac{\partial \ln\left(\prod_{n=1}^{N} f\left(\sum_{j=1}^{N}\omega_{nj}(y_j - \mu s_j);\Phi\right)\right)}{\partial \phi_i}\right]$$

$$= E\left[\frac{\partial \ln\left(\prod_{n=1}^{N} f(u_n;\Phi)\right)}{\partial \mu} \frac{\partial \ln\left(\prod_{n=1}^{N} f(u_n;\Phi)\right)}{\partial \phi_i}\right]$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} E\left[\left(\frac{\partial}{\partial \mu}\ln f(u_n;\Phi)\right)\left(\frac{\partial}{\partial \phi_i}\ln f(u_m;\Phi)\right)\right]$$

$$= \sum_{n=1}^{N} E\left[\left(\frac{\partial}{\partial \mu}\ln f(u_n;\Phi)\right)\left(\frac{\partial}{\partial \phi_i}\ln f(u_n;\Phi)\right)\right] \qquad \text{(as } u_n\text{'s are } i.i.d.\text{)}$$

Using $(A.6)$ this becomes

$$E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathbf{f}}{\partial \phi_i}\right)\right]$$

$$= \sum_{n=1}^{N}\left(-\sum_{j=1}^{N}\omega_{nj}s_j\right) E \underbrace{\left[\underbrace{\left(\frac{\partial}{\partial u_n}\ln f(u_n;\Phi)\right)}_{odd}\underbrace{\left(\frac{\partial}{\partial \phi_i}\ln f(u_n;\Phi)\right)}_{even}\right]}_{odd}$$

Under the assumption that f is even, $\ln f$ is even, derivative of $\ln f$ $w.r.t.$ u_n is odd and the derivative of $\ln f$ $w.r.t.$ ϕ_i is even [Kay 1985]. Therefore the expectation is taken on an odd function and should be equal to zero as explained while proving $(A.7)$. This being true for each ϕ_i one can conclude that $(A.2)$ holds. (26) is a direct implication of $(A.1)$ and $(A.2)$.

Appendix B

Asymptotic Optimality of the GLRT for an AR Model of the Noise

It is now shown that (26) also holds in the case of the GLRT given by (20) for detection in AR noise. Note that the *conditional* likelihood function (see (19)) is used and the vector of nuisance parameters is

$$\Theta_s = [\mathbf{a}\ \Phi]$$

with the notations used before. Proving (26) in this case is equivalent to proving that

$$I_{\mu\mathbf{a}}(\mu, \mathbf{a}, \Phi) = E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathbf{f}}{\delta \mathbf{a}}\right)^T\right] = 0 \tag{$B.1$}$$

and

$$I_{\mu\Phi}(\mu, \mathbf{a}, \Phi) = E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathbf{f}}{\delta \Phi}\right)^T\right] = 0 \tag{$B.2$}$$

To prove $(B.1)$ it is observed that

$$E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathbf{f}}{\partial a_i}\right)\right]$$

$$= E\left[\frac{\partial \ln\left(\prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} a_j(y_j - \mu s_j); \Phi\right)\right)}{\partial \mu} \frac{\partial \ln\left(\prod_{n=p+1}^{N} f\left(\sum_{j=0}^{p} a_j(y_j - \mu s_j); \Phi\right)\right)}{\partial a_i}\right].$$

(16) can be used to simplify the argument of f,

$$E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathbf{f}}{\partial a_i}\right)\right] = E\left[\frac{\partial \ln\left(\prod_{n=p+1}^{N} f(u_n; \Phi)\right)}{\partial \mu} \frac{\partial \ln\left(\prod_{n=p+1}^{N} f(u_n; \Phi)\right)}{\partial a_i}\right]$$

$$= \sum_{n=p+1}^{N} \sum_{m=p+1}^{N} E\left[\left(\frac{\partial}{\partial \mu} \ln f(u_n; \Phi)\right)\left(\frac{\partial}{\partial a_i} \ln f(u_m; \Phi)\right)\right]$$

$$= \sum_{n=p+1}^{N} E\left[\left(\frac{\partial}{\partial \mu} \ln f(u_n; \Phi)\right)\left(\frac{\partial}{\partial a_i} \ln f(u_n; \Phi)\right)\right] \tag{$B.3$}$$

The last step follows from $(A.5)$ and the fact that u_n's are iid. The derivatives *w.r.t.* μ and a_i can be written in terms of the derivative *w.r.t.* u_n. From $(16b)$ it follows that

$$\frac{\partial u_n}{\partial \mu} = -\sum_{j=0}^{p} a_j s_{n-j} \tag{B.4}$$

and

$$\frac{\partial u_n}{\partial a_i} = (y_{n-i} - \mu s_{n-j}) \tag{B.5}$$

Using these results $(B.3)$ can be rewritten as

$$E\left[\left(\frac{\partial \ln \mathbf{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathbf{f}}{\partial \phi_i}\right)\right]$$

$$= \sum_{n=p+1}^{N} E\left[\left(\frac{\partial \ln f}{\partial u_n}\right)\left(\frac{\partial u_n}{\partial \mu}\right)\left(\frac{\partial \ln f}{\partial u_n}\right)\left(\frac{\partial u_n}{\partial a_i}\right)\right]$$

$$= \sum_{n=p+1}^{N}\left(-\sum_{j=0}^{p} a_j s_{n-j}\right) E \underbrace{\left[\underbrace{\left[\frac{\partial}{\partial u_n}\ln f(u_n;\Phi)\right]^2}_{even}\underbrace{\left[y_{n-i} - \mu s_{n-i}\right]}_{odd}\right]}_{odd}$$

$(y_{n-i} - \mu s_{n-i})$ is a linear function of $\{u_{n-i}, u_{n-i-1}, \cdots, u_1\}$ and hence is an odd function of each u_n. Therefore the expectation is taken on an odd function which shoud be equal to zero since the PDF f itself is even. This being true for each a_i, it can be concluded that $(B.1)$ holds.

Proof of $(B.2)$ is similar. Consider

$$E\left[\left(\frac{\partial \ln \mathrm{f}}{\partial \mu}\right)\left(\frac{\partial \ln \mathrm{f}}{\partial \phi_i}\right)\right]$$

$$= E\left[\frac{\partial \ln\left(\prod_{n=p+1}^{N} f(u_n;\Phi)\right)}{\partial \mu}\;\frac{\partial \ln\left(\prod_{n=p+1}^{N} f(u_n;\Phi)\right)}{\partial \phi_i}\right]$$

$$= \sum_{n=p+1}^{N}\sum_{m=p+1}^{N} E\left[\left(\frac{\partial}{\partial \mu}\ln f(u_n;\Phi)\right)\left(\frac{\partial}{\partial \phi_i}\ln f(u_m;\Phi)\right)\right]$$

$$= \sum_{n=p+1}^{N} E\left[\left(\frac{\partial}{\partial \mu}\ln f(u_n;\Phi)\right)\left(\frac{\partial}{\partial \phi_i}\ln f(u_n;\Phi)\right)\right] \qquad \text{(as } u_n\text{'s are iid)}$$

$$= \sum_{n=p+1}^{N}\left(-\sum_{j=0}^{p} a_j s_{n-j}\right) E\underbrace{\left[\underbrace{\left(\frac{\partial}{\partial u_n}\ln f(u_n;\Phi)\right)}_{odd}\underbrace{\left(\frac{\partial}{\partial \phi_i}\ln f(u_n;\Phi)\right)}_{even}\right]}_{odd}$$

(using $(B.4)$)

Since f is even, derivative of $\ln f$ *w.r.t.* u_n is odd and that *w.r.t.* ϕ_i is even. The expectation is therefore taken on and odd function and must equal zero. Since this is true for each ϕ_i, it can be concluded that $(B.2)$ holds. Consequently, (26) holds for the case of AR noise when the GLRT is computed on the basis of conditional likelihood function as in (20).

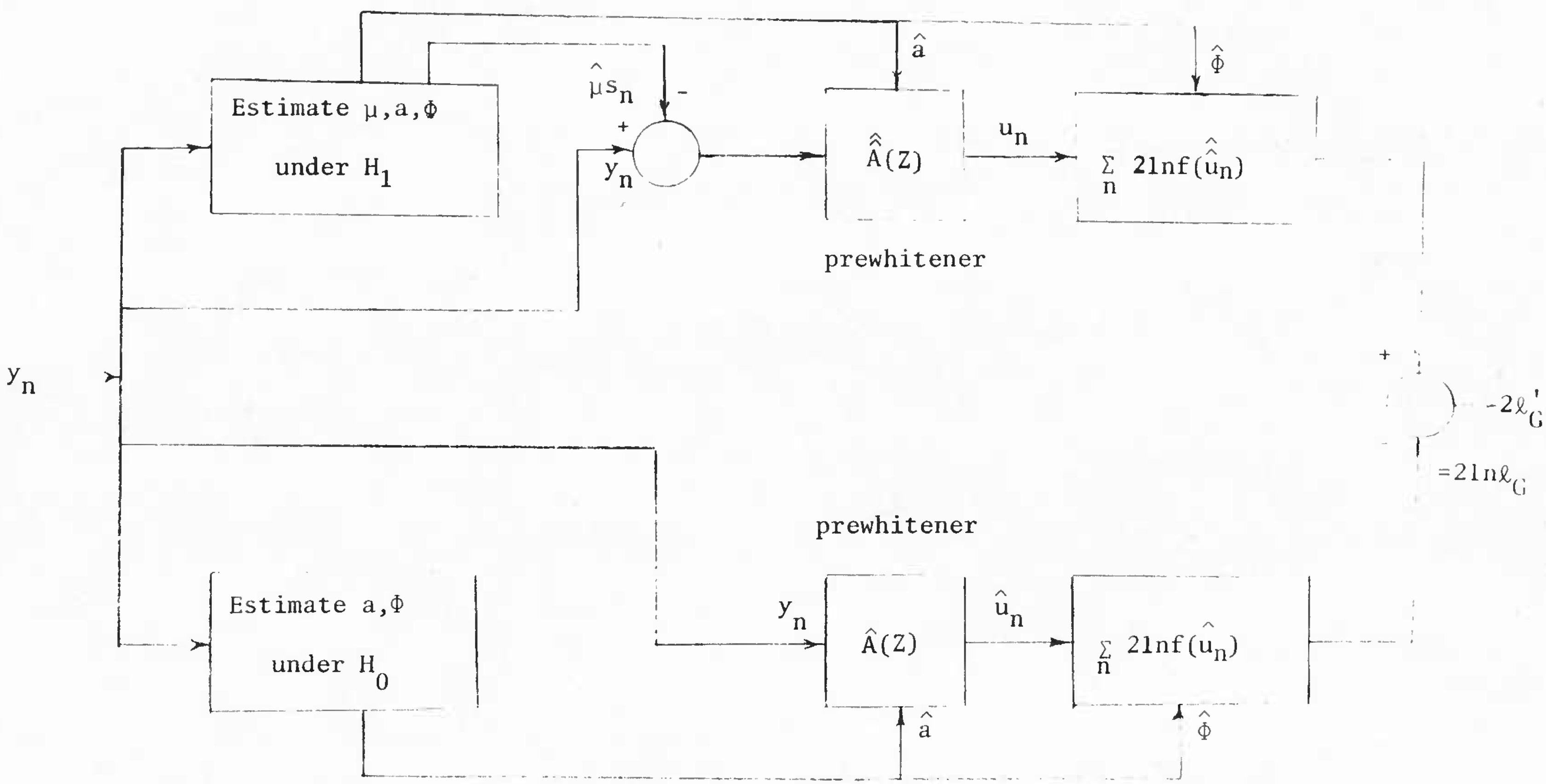

Figure 1 Block diagram of GLRT

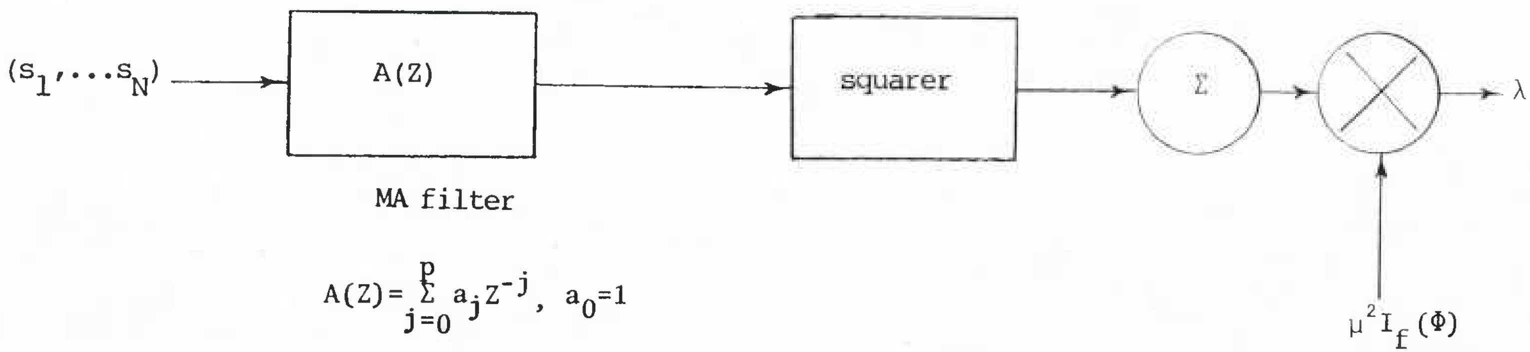

Figure 2 Composition of λ

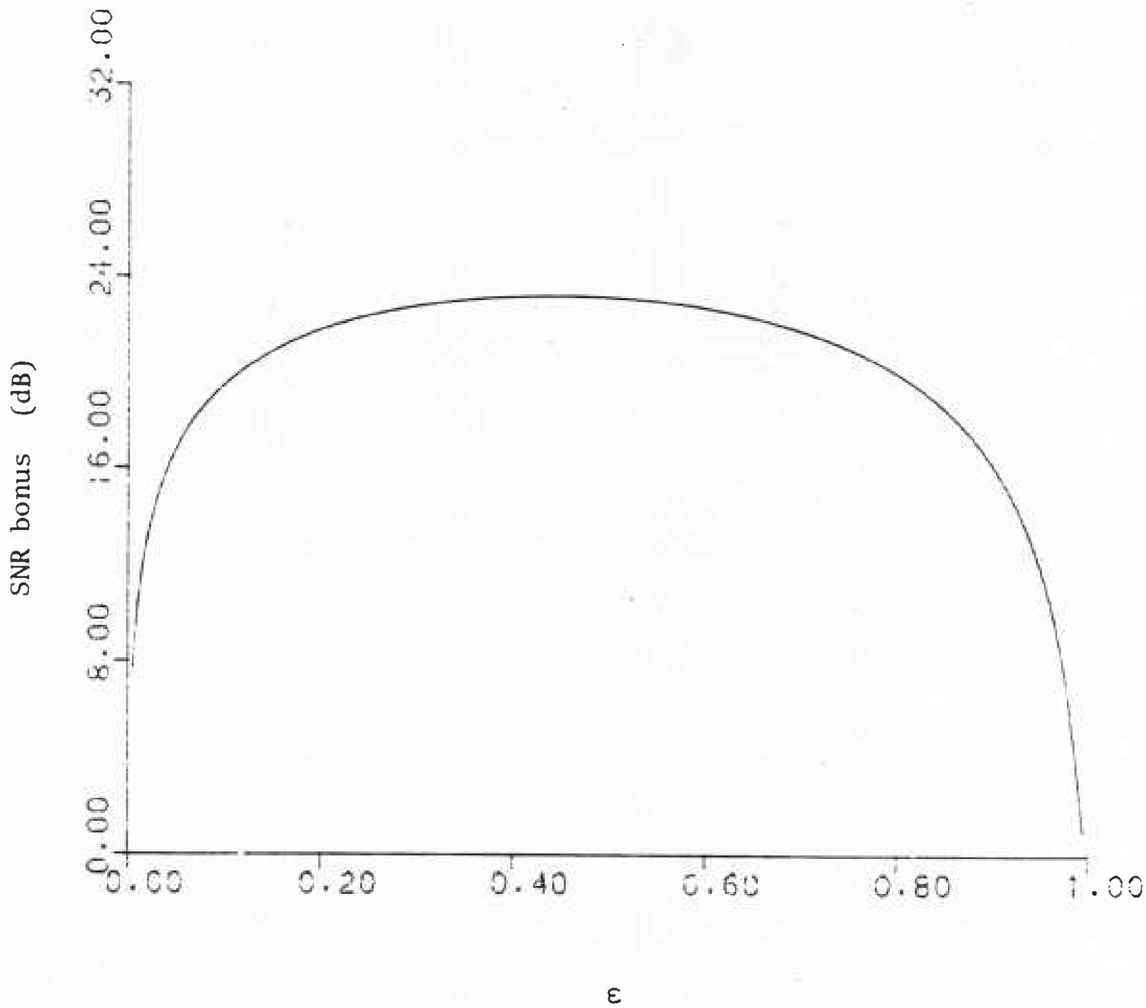

Figure 3 SNR bonus vs. ε
for a Mixed-Gaussian process

OFFICE OF NAVAL RESEARCH
STATISTICS AND PROBABILITY PROGRAM

BASIC DISTRIBUTION LIST
FOR
UNCLASSIFIED TECHNICAL REPORTS

FEBRUARY 1982

	Copies
Statistics and Probability Program (Code 411(SP)) Office of Naval Research Arlington, VA 22217	3
Defense Technical Information Center Cameron Station Alexandria, VA 22314	12
Commanding Officer Office of Naval Research Eastern/Central Regional Office Attn: Director for Science Barnes Building 495 Summer Street Boston, MA 02210	1
Commanding Officer Office of Naval Research Western Regional Office Attn: Dr. Richard Lau 1030 East Green Street Pasadena, CA 91101	1
U. S. ONR Liaison Office - Far East Attn: Scientific Director APO San Francisco 96503	1
Applied Mathematics Laboratory David Taylor Naval Ship Research and Development Center Attn: Mr. G. H. Gleissner Bethesda, Maryland 20084	1
Commandant of the Marine Coprs (Code AX) Attn: Dr. A. L. Slafkosky Scientific Advisor Washington, DC 20380	1
Navy Library National Space Technology Laboratory Attn: Navy Librarian Bay St. Louis, MS 39522	1
U. S. Army Research Office P.O. Box 12211 Attn: Dr. J. Chandra Research Triangle Park, NC 27706	1
Director National Security Agency Attn: R51, Dr. Maar Fort Meade, MD 20755	1
ATAA-SL, Library U.S. Army TRADOC Systems Analysis Activity Department of the Army White Sands Missile Range, NM 88002	1
ARI Field Unit-USAREUR Attn: Library c/o ODCSPER HQ USAEREUR & 7th Army APO New York 09403	1
Library, Code 1424 Naval Postgraduate School Monterey, CA 93940	1
Technical Information Division Naval Research Laboratory Washington, DC 20375	1
OASD (I&L), Pentagon Attn: Mr. Charles S. Smith Washington, DC 20301	1

Copies

Director
AMSAA
Attn: DRXSY-MP, H. Cohen
Aberdeen Proving Ground, MD 21005 — 1

Dr. Gerhard Heiche
Naval Air Systems Command (NAIR 03)
Jefferson Plaza No. 1
Arlington, VA 20360 — 1

Dr. Barbara Bailar
Associate Director, Statistical Standards
Bureau of Census
Washington, DC 20233 — 1

Leon Slavin
Naval Sea Systems Command (NSEA 05H)
Crystal Mall #4, Rm. 129
Washington, DC 20036 — 1

B. E. Clark
RR #2, Box 647-B
Graham, NC 27253 — 1

Naval Underwater Systems Center
Attn: Dr. Derrill J. Bordelon
Code 601
Newport, Rhode Island 02840 — 1

Naval Coastal Systems Center
Code 741
Attn: Mr. C. M. Bennett
Panama City, FL 32401 — 1

Naval Electronic Systems Command (NELEX 612)
Attn: John Schuster
National Center No. 1
Arlington, VA 20360 — 1

Defense Logistics Studies Information Exchange
Army Logistics Management Center
Attn: Mr. J. Dowling
Fort Lee, VA 23801 — 1

Copies

Reliability Analysis Center (RAC)
RADC/RBRAC
Attn: I. L. Krulac
Data Coordinator/
Government Programs
Griffiss AFB, New York 13441 — 1

Technical Library
Naval Ordnance Station
Indian Head, MD 20640 — 1

Library
Naval Ocean Systems Center
San Diego, CA 92152 — 1

Technical Library
Bureau of Naval Personnel
Department of the Navy
Washington, DC 20370 — 1

Mr. Dan Leonard
Code 8105
Naval Ocean Systems Center
San Diego, CA 92152 — 1

Dr. Alan F. Petty
Code 7930
Naval Research Laboratory
Washington, DC 20375 — 1

Dr. M. J. Fischer
Defense Communications Agency
Defense Communications Engineering Center
1860 Wiehle Avenue
Reston, VA 22090 — 1

Mr. Jim Gates
Code 9211
Fleet Material Support Office
U. S. Navy Supply Center
Mechanicsburg, PA 17055 — 1

Mr. Ted Tupper
Code M-311C
Military Sealift Command
Department of the Navy
Washington, DC 20390 — 1

Copies

Mr. F. R. Del Priori
Code 224
Operational Test and Evaluation
Force (OPTEVFOR)
Norfolk, VA 23511 1

Copies